11.19 Linear Transformation by 姜姜

Section1 Linear Transformation, Kernel and Image, Dimension Thm

Prob.1.1 Let V and W be vector spaces and T: V→W be linear.
(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.
(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if T(S) is linearly independent.
(c) Suppose $\beta=\{v_1, v_2,\ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta)=\{T(v_1), T(v_2),\ldots, T(v_n)\}$ is a basis for W.

Def. Let V be a vector space and W_1 and W_2 be subspaces of V such that $V=W_1\oplus W_2$. A function T: V→V is called the projection on W_1 along W_2 if, for $x=x_1+x_2$ with $x_1\in W_1$ and $x_2\in W_2$, we have $T(x)=x_1$.
Prob.1.2 Using the notion in the definition above, assume that T: V→V is the projection on W_1 along W_2.
(a) Prove that T is linear and $W_1=\{x\in V: T(x)=x\}$.
(b) Prove that $W_1=\mathrm{Im}(T)$ and $W_2=\mathrm{Ker}(T)$.
(c) Describe T if $W_1=V$.
(d) Describe T if W_1 is the zero subspace.

Prob.1.3 Suppose that W is a subspace of a finite-dimensional vector space V.
(a) Prove that there exists a subspace W' and a function T: V→V such that T is a projection on W along W'.
(b) Given an example of a subspace W of a vector space V such that there are two projections on W along two (distinct) subspace.

Prob.1.4 Let V be a finite-dimensional vector space and T: V→V be linear.
(a) Suppose that V=Im(T)+Ker(T). Prove that $V=\mathrm{Im}(T)\oplus\mathrm{Ker}(T)$.
(b) Suppose that $\mathrm{Im}(T)\cap\mathrm{Ker}(T)=\{0\}$. Prove that $V=\mathrm{Im}(T)\oplus\mathrm{Ker}(T)$.
Be careful to say in each part where finite-dimensionality is used.

Section2 The Matrix Representation of a Linear Map

Prob.2.1 Let V and W be vector spaces, and let S be a subset of V. Define $S^0=\{T\in \mathcal{L}(V, W):T(x)=0$ for all $x\in S\}$. Prove the following statements.
(a) S^0 is a subspace of $\mathcal{L}(V, W)$.
(b) If S_1 and S_2 are subsets of V and $S_1\subseteq S_2$, then $S_2^0\subseteq S_1^0$.
(c) If V_1 and V_2 are subspaces of V, then $(V_1+V_2)^0=V_1^0\cap V_2^0$.

Prob.2.2 Let V and W be vector spaces such that dim(V)=dim(W),and let T: V→W be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Section3 Composition of Linear Maps and Matrix Multiplication

Prob.3.1 Let A be an m×n matrix and B be an n×p matrix. For each j (1≤j≤p) let u_j and v_j denote the jth columns of AB and B, respectively.(这题小盆友们自己看一下哦，我不讲啦~

(a) Suppose that z is a (column) vector in F^p. Prove that Bz is a linear combination of the columns of B. In particular, if $z=(a_1, a_2,\dots, a_p)^T$, then show that Bz= $\sum_{j=1}^{p} aj{\cdot}vj$

(b) Extend (a) to prove that column j of AB is a linear combination of the columns of A with the coefficients in the linear combination being the entries of column of B.

(c) For any row vector w ∈ F^m, prove that wA is a linear combination of the rows of A with the coefficients in the linear combination being the coordinates of w. Hint: Use properties of the transpose operation applied to (a).

(d) Prove the analogous result to (b) about rows: Row i of AB is a linear combination of the rows of B with the coefficients in the linear combination being the entries of row i of A.

14. Proof. (a) We have

$$Bz = B(a_1e_1 + \cdots + a_pe_p) = a_1Be_1 + \cdots + a_pBe_p = a_1v_1 + \cdots + a_pv_p.$$

(b) Applying (a), the column j of AB is

$$u_j = Av_j = A(b_{1j}e_1 + \cdots + b_{nj}e_n) = b_{1j}y_1 + \cdots + b_{nj}y_n,$$

where $y_1, \dots, y_n$ are the columns of A.

(c) Write $w = (a_1, \dots, a_m)$. Applying (a) for A^T and w^T, we see that

$$A^T w^T = a_1x_1 + \cdots + a_mx_m,$$

where $x_1, \dots, x_m$ are the columns of A^T. By definition of transpose, $x_1^T, \dots, x_m^T$ are the rows of A. Taking transpose of the above equation gives that

$$wA = a_1x_1^T + \cdots a_mx_m^T,$$

which is the required assertion.

(d) By (b), column i of $(AB)^T = B^TA^T$ is the linear combination of columns of B^T with coefficients being $(A^T)_{1i} = a_{i1}, \dots, (A^T)_{ni} = a_{in}$. Taking transpose, row i of AB is the linear combination of rows of B with coefficients being $a_{i1}, \dots, a_{in}$, which is the required assertion.

Prob.3.2 Let V be a finite-dimensional vector space, and let T: V→V be linear.

(a) If rank(T)=rank(T^2), prove that Im(T)∩Ker(T)={0}. Deduce that V=Im(T)⊕Ker(T).

(b) Prove that V=Im(T^k)⊕Ker(T^k) for some positive integer k.

Prob.3.3 Let V be a vector space. Determine all linear transformations T: V→V such that T=T^2.
Hint: Note that x=T(x)+(x-T(x)) for every x in V, and show that V={y:T(y)=y}⊕Ker(T).

Section4 Invertibility and Isomorphism

Prob.4.1 Let V and W be n-dimensional vectorspaces, and let T: V→W be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if T(β)is a basis for W.

Prob.4.2 Let V and W be finite-dimensional vector spaces and T: V→W be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim(V_0)=\dim(T(V_0))$.

Prob.4.3 Let T: V→W be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $\text{rank}(T)=\text{rank}(L_A)$ and $\dim\text{Ker}(T)=\dim\text{Ker}(L_A)$, where $A=[T]_\beta^\gamma$.

Thm Let V and W be vector spaces over F,and suppose that $\{v_1, v_2,\dots, v_n\}$ is a basis for V. For $w_1, w_2,\dots, w_m$ in W, there exists exactly one linear transformation T: V→W such that $T(v_i)=w_i$ for $i=1,2,\dots, n$.

Prob.4.4 Let V and W be finite-dimensional vector spaces with ordered bases $\beta=\{v_1, v_2,\dots, v_n\}$ and $\gamma=\{w_1, w_2,\dots, w_m\}$, respectively. By thm, there exist linear transformations T_{ij}: V→W such that $T_{ij}(vk)$ $= \begin{cases} wi & if\ k=j \\ 0 & if\ k\neq j \end{cases}$

First prove that $\{T_{ij}: 1\le i\le m, 1\le j\le n\}$ is a basis for L(V, W). Then let M^{ij} be the m×n matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[T_{ij}]_\beta^\gamma=M^{ij}$.Again by thm, there exists a linear transformation Φ: L(V, W)→$M_{m\times n}(F)$ such that $\Phi(T_{ij})=M^{ij}$. Prove that Φ is an isomorphism. Hint: you may use Prob4.1.