一、增广矩阵 $\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 & -3 & a \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 5 & 4 & 3 & 3 & -1 & 2 \end{array}\right) \xrightarrow{\text{行变换}} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $a\neq 0$ 无解

$a=0$ 有解.

二、$\{2\alpha_1+\alpha_2, 2\alpha_2+\alpha_3, 2\alpha_3+\alpha_1\}$ 线性无关 $\Longleftrightarrow$ $\{\alpha_1,\alpha_2,\alpha_3\}$ 线性无关

Pf: "$\Rightarrow$" 设 $\begin{cases} \beta_1=2\alpha_1+\alpha_2 \\ \beta_2=2\alpha_2+\alpha_3 \\ \beta_3=2\alpha_3+\alpha_1 \end{cases} \rightsquigarrow \begin{cases} \alpha_1=\frac{1}{9}(4\beta_1-2\beta_2+\beta_3) \\ \alpha_2=\frac{1}{9}(\beta_1+4\beta_2-2\beta_3) \\ \alpha_3=\frac{1}{9}(-2\beta_1+\beta_2+4\beta_3) \end{cases}$

设 $k_1\alpha_1+k_2\alpha_2+k_3\alpha_3=\frac{1}{9}\left[(4k_1+k_2-2k_3)\beta_1+(-2k_1+4k_2+k_3)\beta_2+(k_1-2k_2+4k_3)\beta_3\right]=0$

$\{\beta_1,\beta_2,\beta_3\}$ 线性无关 $\Rightarrow \begin{cases} 4k_1+k_2-2k_3=0 \\ -2k_1+4k_2+k_3=0 \\ k_1-2k_2+4k_3=0 \end{cases} \Rightarrow k_1=k_2=k_3=0 \Rightarrow \{\alpha_1,\alpha_2,\alpha_3\}$ 线性无关

"$\Leftarrow$" 设 $k_1(2\alpha_1+\alpha_2)+k_2(2\alpha_2+\alpha_3)+k_3(2\alpha_3+\alpha_1)=(2k_1+k_3)\alpha_1+(2k_2+k_1)\alpha_2+(2k_3+k_2)\alpha_3=0.$

$\{\alpha_1,\alpha_2,\alpha_3\}$ 线性无关 $\Rightarrow \begin{cases} 2k_1+k_3=0 \\ 2k_2+k_1=0 \\ 2k_3+k_2=0 \end{cases} \Rightarrow k_1=k_2=k_3=0 \Rightarrow \{2\alpha_1+\alpha_2, 2\alpha_2+\alpha_3, 2\alpha_3+\alpha_1\}$ 线性无关

Rmk (一般的做法)

$\{\alpha_1\cdots\alpha_n\}$ 线性无关 $A: n\times s$ 矩阵 且有 $(\beta_1\cdots\beta_s)=(\alpha_1\cdots\alpha_n)A$ 则 $\mathrm{rank}\{\beta_1\cdots\beta_s\}=\mathrm{rank}(A)$

Pf: 定义 $\sigma: \mathbb{R}^n \to L(\alpha_1\cdots\alpha_n)$ $\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \mapsto \sum_{i=1}^{n} k_i\alpha_i$ σ是同构 因为 σ满 $\dim L(\alpha_1\cdots\alpha_n) = \dim \mathbb{R}^n = n$

设A的第j列为 X_j $(1\leq j\leq s)$ $X_{k_1}\cdots X_{k_r}$ 是A的列向量的极大线性无关组

则 $\sigma(X_j)=\beta_j$ σ同构 $\Rightarrow$ $\beta_{k_1}\cdots\beta_{k_r}$ 是 $\{\beta_1\cdots\beta_s\}$ 的极大线性无关组

所以 $\mathrm{rank}\{\beta_1\cdots\beta_s\}=r=r(A)$

推论 A可逆 $n\times n$ $(\beta_1\cdots\beta_n)=(\alpha_1\cdots\alpha_n)A$ 则 $\{\alpha_1\cdots\alpha_n\}$线性无关 $\Leftrightarrow$ $\{\beta_1\cdots\beta_n\}$线性无关

pf: $\Rightarrow$ $\operatorname{rank}\{\beta_1\cdots\beta_n\}=r(A)=n \rightsquigarrow \{\beta_1\cdots\beta_n\}$线性无关

$\Leftarrow$ $(\alpha_1\cdots\alpha_n)=(\beta_1\cdots\beta_n)A^{-1} \rightsquigarrow \operatorname{rank}\{\alpha_1\cdots\alpha_n\}=r(A^{-1})=n \rightsquigarrow \{\alpha_1\cdots\alpha_n\}$线性无关.

三、$$(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \beta)=\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 4\\ 2 & -1 & -1 & 1 & 2\\ 4 & -6 & 2 & -2 & 4\\ 3 & 6 & -9 & 7 & a\end{array}\right)\xrightarrow{\text{行变换}}\left(\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 4\\ 0 & -3 & 3 & -1 & -6\\ 0 & 0 & 0 & -\frac{8}{3} & 8\\ 0 & 0 & 0 & 0 & a-9\end{array}\right)$$
$$\quad\uparrow\quad\uparrow\qquad\quad\uparrow$$

(1) $\dim W=\operatorname{rank}\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}=3$ 且 $\{\alpha_1\ \alpha_2\ \alpha_4\}$是基

(2) $\beta\in W \Leftrightarrow \beta=x_1\alpha_1+x_2\alpha_2+x_3\alpha_3+x_4\alpha_4$ 有解 $\Leftrightarrow a=9$

由于 $$\begin{pmatrix}4\\-6\\8\\0\end{pmatrix}=4\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+3\begin{pmatrix}1\\-3\\0\\0\end{pmatrix}-3\begin{pmatrix}1\\-1\\-\frac{8}{3}\\0\end{pmatrix}$$ 行变换又不改变列的线性关系

所以 $\beta=4\alpha_1+3\alpha_2-3\alpha_4$

四. (1) $\langle\alpha_1,\alpha_2\rangle=\arccos\dfrac{(\alpha_1,\alpha_2)}{|\alpha_1|\,|\alpha_2|}=\arccos\dfrac{4}{2\sqrt{20}}=\arccos\dfrac{\sqrt5}{5}$

(2) Schimdt 正交化:

$\beta_1=\alpha_1=(1,1,1,1)$

$\beta_2=\alpha_2-\dfrac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=(2,2,-2,-2)$

$\beta_3=\alpha_3-\dfrac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2-\dfrac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1=(-1,1,-1,1)$

$\varepsilon_1=\dfrac{\beta_1}{|\beta_1|}=\left(\frac12,\frac12,\frac12,\frac12\right)$

$\varepsilon_2=\dfrac{\beta_2}{|\beta_2|}=\left(\frac12,\frac12,-\frac12,-\frac12\right)$

$\varepsilon_3=\dfrac{\beta_3}{|\beta_3|}=\left(-\frac12,\frac12,-\frac12,\frac12\right)$

五、(1) 设 $(\sigma(f_1),\sigma(f_2),\sigma(f_3)) = (f_1,f_2,f_3)A$

$$\left.\begin{aligned}(\sigma(f_1),\sigma(f_2),\sigma(f_3)) &= (1,x,x^2)\begin{pmatrix}2&0&1\\0&1&1\\2&0&1\end{pmatrix}\\ (f_1,f_2,f_3)A &= (1,x,x^2)\begin{pmatrix}1&1&0\\-1&0&1\\0&1&2\end{pmatrix}A\end{aligned}\right\} \Rightarrow A = \begin{pmatrix}1&1&0\\-1&0&1\\0&1&2\end{pmatrix}^{-1}\begin{pmatrix}2&0&1\\0&1&1\\2&0&1\end{pmatrix}$$

$$= \begin{pmatrix}-1&-2&1\\2&2&-1\\-1&-1&1\end{pmatrix}\begin{pmatrix}2&0&1\\0&1&1\\2&0&1\end{pmatrix} = \begin{pmatrix}-1&-2&-2\\3&2&3\\-1&-1&-1\end{pmatrix}$$

(或者用待定系数求出
$$\left.\begin{aligned}\sigma(f_1) &= -f_1+3f_2-f_3\\ \sigma(f_2) &= -2f_1+2f_2-f_3\\ \sigma(f_3) &= -2f_1+3f_2-f_1\end{aligned}\right\} \Rightarrow (\sigma(f_1),\sigma(f_2),\sigma(f_3)) = (f_1\ f_2\ f_3)\begin{pmatrix}-1&-2&-2\\3&2&3\\-1&-1&-1\end{pmatrix}$$
)

(2) $$\sigma(f) = \sigma\left((1\ x\ x^2)\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = \sigma\left((f_1\ f_2\ f_3)\begin{pmatrix}1&1&0\\-1&0&1\\0&1&2\end{pmatrix}^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = (\sigma(f_1)\ \sigma(f_2)\ \sigma(f_3))\begin{pmatrix}1&1&0\\-1&0&1\\0&1&2\end{pmatrix}^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$= (1\ x\ x^2)\begin{pmatrix}2&0&1\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\-1&0&1\\0&1&2\end{pmatrix}^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix} = (1\ x\ x^2)\begin{pmatrix}-4\\3\\2\end{pmatrix} = -4+3x+2x^2$$

(或者用待定系数求出 $f = -2f_1+3f_2$

$\sigma(f) = -2\sigma(f_1)+3\sigma(f_2) = -2x^2+3x-4$)

六 $(\sigma+\tau)(x_1,x_2,x_3) = (x_1+3x_2+3x_3,\ -x_1+2x_2,\ 0) = x_1\cdot(1,-1,0)+x_2(3,2,0)+x_3\cdot(3,0,0)$

$\sigma\tau(x_1,x_2,x_3) = \sigma(x_2,x_3,0) = (x_2+2x_3,\ -x_2+2x_3,\ 0) = x_2\cdot(1,-1,0)+x_3\cdot(2,2,0)$

$\mathrm{Im}(\sigma+\tau) = \{x_1\cdot(1,-1,0)+x_2(3,-2,0)+x_3\cdot(3,0,0) \mid x_1,x_2,x_3\in R\} = L((1,-1,0),(3,2,0),(3,0,0))$

$\mathrm{Im}\,\sigma\tau = L((1,-1,0),(2,2,0))$

$r(\sigma+\tau)=\operatorname{rank}\big((1,-1,0),(3,2,0),(3,0,0)\big)=2$ （因为 $\begin{pmatrix}1&3&3\\-1&2&0\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&3&3\\0&5&3\\0&0&0\end{pmatrix}$ ）

$r(\sigma\tau)=\operatorname{rank}\big((1,-1,0),(2,2,0)\big)=2$

其实也可以这样

设 $\sigma:V\to W$ 线性

$B_V=\{\alpha_1\cdots\alpha_n\}$

$\operatorname{im}\sigma=L(\sigma(\alpha_1)\cdots\sigma(\alpha_n))$

$r(\sigma)=\operatorname{rank}\{\sigma(\alpha_1)\cdots\sigma(\alpha_n)\}$

$=\operatorname{rank}(m(\sigma))$

(2) $(x_1+2x_2+3x_3,-x_1+2x_2-x_3,0)=x_1\cdot(1,-1,0)+x_2\cdot(2,2,0)+x_3\cdot(3,-1,0)$

$\operatorname{Im}\sigma=L\big((1,-1,0),(2,2,0),(3,-1,0)\big)$

$\sigma(x_1,x_2,x_3)=0\Rightarrow\begin{cases}x_1+2x_2+3x_3=0\\-x_1+2x_2-x_3=0\end{cases}\Rightarrow\begin{cases}x_1=-2x_3\\x_2=-\frac{1}{2}x_3\end{cases}\Rightarrow\ker\sigma=L\big((-2,-\tfrac{1}{2},1)\big)$

$\operatorname{Im}\sigma+\ker\sigma=L\big((1,-1,0),(2,2,0),(3,-1,0),(-2,-\tfrac{1}{2},1)\big)=R^3$ （因为 $\begin{pmatrix}1&2&3&-2\\-1&2&-1&-\frac{1}{2}\\0&0&0&1\end{pmatrix}\to\begin{pmatrix}1&2&3&-2\\0&4&2&-2.5\\0&0&0&1\end{pmatrix}$ ）

七：设 E_{ij} 为第 i 行和第 j 列的元素为1其他元素全为0的 $n\times n$ 阶矩阵

① $k=0$　$a_{ji}=0$　$i\leq j\Rightarrow W$ 为主对角线全为0的上三角矩阵全体 $\forall A\in W$ $A=\sum_{i=1}^{n}\sum_{j=k}^{n}a_{ij}E_{ij}=\sum_{i>j}a_{ij}E_{ij}$

而 $B_1=\{E_{ij}\mid i>j\}$ 又线性无关，$B_1\subseteq W$ 故 B_1 是 W 的基 $\dim W=\frac{n(n-1)}{2}$

② $k=1$　$a_{ji}=a_{ij}$　$i\leq j\Rightarrow W$ 为对称矩阵全体 $\forall A\in W$ $A=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}E_{ij}=\sum_{i<j}a_{ij}(E_{ij}+E_{ji})+\sum_{i=1}^{n}a_{ii}E_{ii}$

而 $B_2=\{E_{ij}+E_{ji},E_{kk}\mid 1\leq i,j,k\leq n\ \ i<j\}$ 又线性无关，$B_2\subseteq W$ 故 B_2 是 W 的基 $\dim W=\frac{n(n-1)}{2}+n=\frac{n(n+1)}{2}$

③ $k=2$　$a_{ii}=2a_{ii}\ (1\leq i\leq n)\Rightarrow a_{ii}=0$　$\forall A\in W$ $A=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}E_{ij}=\sum_{i>j}(2E_{ij}+E_{ji})$

而 $B_3=\{2E_{ij}+E_{ji}\mid i>j\}$ 又线性无关 $B_3\subseteq W$ 故 B_3 是 W 的基 $\dim W=\frac{n(n-1)}{2}$

八 (1) $k_1=k_2=\cdots=k_n=0 \Rightarrow 0\in V_2 \Rightarrow V_2\neq\phi$

$\forall \xi,\eta\in V_2$ 设 $\xi=\sum_{i=1}^{n} s_i\alpha_i \quad \sum_{i=1}^{n}\frac{s_i}{i}=0$ $\quad \forall k,l\in F \quad k\xi+l\eta=\sum_{i=1}^{n}(ks_i+lt_i)\alpha_i$ 由于 $\sum_{i=1}^{n}\frac{ks_i+lt_i}{i}=k\cdot\sum_{i=1}^{n}\frac{s_i}{i}+l\cdot\sum_{i=1}^{n}\frac{t_i}{i}$

$\eta=\sum_{i=1}^{n} t_i\alpha_i \quad \sum_{i=1}^{n}\frac{t_i}{i}=0$ $\quad$ 故 $k\xi+l\eta\in V_2$ 所以 V_2 是子空间 $\quad =k\cdot 0+l\cdot 0=0$

(2) 设 $\xi\in V_1\cap V_2$

$\xi\in V_1 \Rightarrow \exists\lambda$ s.t. $\xi=\lambda(\alpha_1+2\alpha_2+\cdots n\alpha_n)=\lambda\alpha_1+2\lambda\alpha_2+\cdots n\lambda\alpha_n$

$\xi\in V_2 \Rightarrow \lambda+\frac{2\lambda}{2}+\cdots+\frac{n\lambda}{n}=n\lambda=0 \Rightarrow \lambda=0. \Rightarrow \xi=0 \Rightarrow V_1\cap V_2=\{0\}.$

$\forall\eta\in V$ 设 $\eta=\sum_{i=1}^{n}\lambda_i\alpha_i$ 设 $t=\frac{1}{n}\left(\lambda_1+\frac{\lambda_2}{2}+\cdots\cdots+\frac{\lambda_n}{n}\right)$

$\sum_{i=1}^{n}\frac{\lambda_i-it}{i}=\sum_{i=1}^{n}\left(\frac{\lambda_i}{i}-t\right)=\sum_{i=1}^{n}\frac{\lambda_i}{i}-nt=0 \rightsquigarrow \sum_{i=1}^{n}\lambda_i\alpha_i-t(\alpha_1+2\alpha_2+\cdots n\alpha_n)\in V_2$

$\eta=\sum_{i=1}^{n}\lambda_i\alpha_i=\underbrace{t(\alpha_1+2\alpha_2+\cdots n\alpha_n)}_{\in V_1}+\underbrace{\sum_{i=1}^{n}\lambda_i\alpha_i-t(\alpha_1+2\alpha_2+\cdots n\alpha_n)}_{\in V_2} \Rightarrow V\subseteq V_1+V_2$

V_1+V_2 显然 $\subseteq V$ 故 $V=V_1+V_2$

$$\left.\begin{array}{l}V=V_1+V_2\\ V_1\cap V_2=\{0\}\end{array}\right\}\Rightarrow V=V_1\oplus V_2$$

九. (1) 正确。假设存在这样的 f，f 非零 $\Rightarrow \exists A \in M_n(\mathbb{R})$ $f(A) \neq 0$ $f(A)=f(AE)=f(A)f(E) \Rightarrow f(E)=1$ E为单位矩阵

设 E_{ij} 为第 i 行和第 j 列的元素为1其他元素全为0的 $n\times n$ 阶矩阵

$\forall 1\leq i\leq n$ $f(E_{ii})=f(E_{ii}^2)=f(E_{ii})^2 \Rightarrow f(E_{ii})=0$ 或 1

$1=f(E)=\sum_{i=1}^{n} f(E_{ii}) \Rightarrow \exists$ 唯一的 $1\leq k\leq n$ s.t. $f(E_{kk})=1$ $f(E_{ll})=0$ $\forall l\neq k$

不妨设 $k=1$ 即 $f(E_{11})=1$ $f(E_{ll})=0$ $\forall l\neq 1$

$f(E_{11})=f(E_{12}E_{21})=f(E_{12})f(E_{21})=f(E_{21})f(E_{12})=f(E_{21}E_{12})=f(E_{22}) \Rightarrow 1=0$ 矛盾 故不存在

(2) 正确。$W_1\cup W_2=W_1+W_2 \Leftrightarrow W_1\subseteq W_2$ 或者 $W_2\subseteq W_1$

解: $\Leftarrow$ 不妨设 $W_1\subseteq W_2$，$W_1\cup W_2=W_2$，$W_1+W_2=W_2$ (因为 $\forall w\in W_2$ $w=0+w\in W_1+W_2$
$\forall w\in W_1+W_2$ $\exists w_1\in W_1$ $w_2\in W_2$ s.t. $w=w_1+w_2$
而 $w_1\in W_1\subseteq W_2$ W_2 关于+封闭 $\Rightarrow w\in W_2$)

故 $W_1\cup W_2=W_1+W_2$

$\Rightarrow$ 如果 $W_1\not\subseteq W_2$ 且 $W_2\not\subseteq W_1$ 则 $\exists \alpha\in W_1$ $\alpha\notin W_2$ $\alpha+\beta\in W_1+W_2=W_1\cup W_2$
$\exists\beta\in W_2$ $\beta\notin W_1$

① $\alpha+\beta\in W_1 \Rightarrow \exists \gamma\in W_1$ $\alpha+\beta=\gamma$ $\beta=\gamma-\alpha\in W_1$ 矛盾

② $\alpha+\beta\in W_2 \Rightarrow \exists \gamma\in W_2$ $\alpha+\beta=\gamma$ $\alpha=\gamma-\beta\in W_2$ 矛盾

(3) 正确 α,β 线性无关 $\Rightarrow \alpha,\beta$ 都不是零向量

设 θ 为 α,β 的夹角 $4\cos^2\theta=4\left(\frac{(\alpha,\beta)}{|\alpha||\beta|}\right)^2=\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}=\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\cdot\frac{2(\alpha,\beta)}{(\beta,\beta)}\in\{0,1,2,3,4\}$.

$\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\leq 0 \Rightarrow (\alpha,\beta)\leq 0 \Rightarrow \cos\theta\leq 0$

$\Rightarrow \cos\theta = 0, -\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}, -1 \quad \theta = \frac{\pi}{2}, \frac{2}{3}\pi, \frac{3}{4}\pi, \frac{5}{6}\pi, \pi$

由α, β线性无关 $\alpha \neq -\beta$ 故$\theta \neq \pi$ 所以 $\theta = \frac{\pi}{2}, \frac{2}{3}\pi, \frac{3}{4}\pi, \frac{5}{6}\pi$

(4) 取 $n=2$ $(i,1) \in W$ $(1,i) \in W$ $(i+1)^2 + (i+1)^2 = 4i \neq 0 \Rightarrow (i+1, i+1) \notin W$

W关于加法不封闭 $\Rightarrow W$不是子空间