一. 重积分

例1: $\int_a^b f(x)dx\cdot\int_a^b \frac{1}{f(x)}dx = \int_a^b f(x)dx\cdot\int_a^b \frac{1}{f(y)}dy$

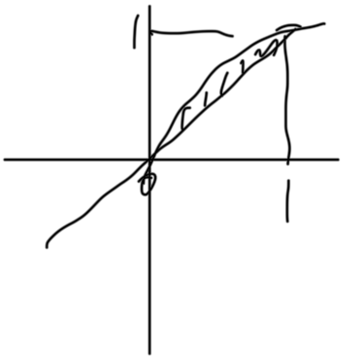
$$= \iint_{[a,b]\times[a,b]} \frac{f(x)}{f(y)}dxdy$$

$$= \iint_{[a,b]\times[a,b]} \frac{f(y)}{f(x)}dxdy$$

$$\geq \frac{1}{2}\iint_{[a,b]\times[a,b]} \left(\frac{f(x)}{f(y)} + \frac{f(y)}{f(x)}\right)dxdy$$

$$= \frac{1}{2}\cdot 2\cdot (b-a)^2 = (b-a)^2$$

例2:



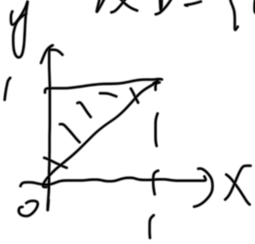
X型做一积不出来

Y型: 原式 $= \int_0^1 dy\int_{y^2}^{y} \frac{\sin y}{y}dx$

$$= \int_0^1 \frac{\sin y}{y}(y-y^2)dy = 1-\sin 1$$

例3: 用二重积分搭析

设 $D=\{(x,y)\in\mathbb{R}^2 \mid x\leq y\leq 1,\ 0\leq x\leq 1\}$, 计算 $I=\iint_D e^{-y^2}dxdy$



X型: 原式

Y型: $I = \int_0^1 \left(\int_0^y e^{-y^2}dx\right)dy$

$$= \int_0^1 e^{-y^2}\cdot y\cdot dy$$

$$= -\frac{1}{2}e^{-y^2}\Big|_0^1 = -\frac{1}{2}\left(\frac{1}{e}-1\right) = \frac{1}{2}-\frac{1}{2e}$$

例4: 由二重积分中值定理, $\exists(\xi,\eta)\in D$, s.t.

$$\frac{1}{\pi r^2}\iint_D e^{x^2-y^2}\cos(x+y)dxdy = e^{\xi^2-\eta^2}\cos(\xi+\eta)$$

$r\to 0^+$, $(\xi,\eta)\to(0,0)$

原式 $=1$

例5：[前情提要] 含参变量积分：$\varphi(x)=\int_\alpha^\beta f(x,y)dy$.

定理1（连续性）若 $f(x,y)$ 在 $R:[a,b]\times[\alpha,\beta]$ 上连续，则 $\varphi(x)$ 在 $[a,b]$ 上连续

定理2. 若 $f(x,y)$ 在 $R=[a,b]\times[c,d]$ 上连续，则：

$$\int_a^b\left[\int_c^d f(x,y)dy\right]dx = \int_c^d\left[\int_a^b f(x,y)dx\right]dy$$

↳ 累次积分可交换顺序

定理3（可微性）. 若 $f(x,y)$ 及其偏导数 $f_x(x,y)$ 在 $R=[a,b]\times[\alpha,\beta]$ 上连续，则 $\varphi(x)=\int_\alpha^\beta f(x,y)dy$ 在 $[a,b]$ 上可微

且 $\varphi'(x)=\dfrac{d}{dx}\int_\alpha^\beta f(x,y)dy=\int_\alpha^\beta f_x(x,y)dy$

回到原题，$\int_a^b x^y dy=\dfrac{x^y}{\ln x}\Big|_a^b=\dfrac{x^b-x^a}{\ln x}$

$I=\int_0^1 dx\int_a^b x^y dy$　　x^y 显然在 $[0,1]\times[a,b]$ 连续，

$$=\int_a^b dy\int_0^1 x^y dx$$

$$=\int_a^b\left(\frac{x^{y+1}}{y+1}\Big|_0^1\right)dy$$

$$=\int_a^b\frac{1}{y+1}dy$$

$$=\ln\frac{b+1}{a+1}$$

二、1. Green公式

Green公式复习：① $P(x,y)$、$Q(x,y)$有一阶连续导函数

② L闭合

③ 正向定义

例1：$\int_{\partial D} \frac{F(xy)}{y} dy = \iint_D \left[\frac{\partial \frac{F(xy)}{y}}{\partial x}\right] dxdy$

$= \iint_D \frac{1}{y} F'(xy) \cdot y \cdot dxdy$

$= \iint_D F'(xy) dxdy$

$= \iint_D f(xy) dxdy$

令 $u = xy$, $v = \frac{y}{x}$，则：$x = \sqrt{\frac{u}{v}}$, $y = \sqrt{uv}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v\sqrt{v}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix} = \frac{1}{2v}$$

$D_{uv} = \{(u,v) \mid 1 \le u \le 4,\ 1 \le v \le 4\}$

$$\int_{\partial D} \frac{F(x,y)}{y} dy = \iint_{D_{uv}} f(u) \frac{1}{2v} dudv = \int_1^4 f(u)du \cdot \int_1^4 \frac{1}{2v} dv$$

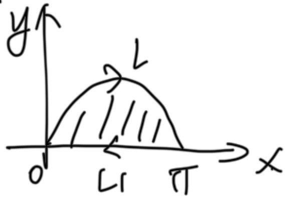
$= \ln 2 \int_1^4 f(u) du$

(2) 代入即可

$$\int_{\partial D} \frac{F(xy)}{y} = \ln 2 \int_1^4 F'(u) du = \ln 2\, F(u) \Big|_1^4$$

$= \ln 2$

例2：



L_1: $y=0$, $x:\pi\to 0$

$$\int_{L+L_1} \underbrace{\sqrt{x^2+y^2}}_{=P}\,dx + \underbrace{y[xy+\ln(x+\sqrt{x^2+y^2})]}_{Q}\,dy$$

$$= - \iint_D \left(y^2+\frac{y}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\right)dxdy$$

方向性

$$= -\iint_D y^2\,dxdy$$

$$= -\frac{4}{9}$$

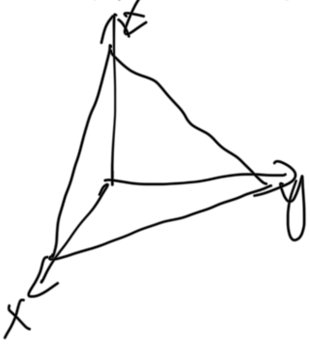
$$\int_{L_1} \sqrt{x^2+y^2}\,dx + y[xy+\ln(x+\sqrt{x^2+y^2})]dy = \int_{L_1} x\,dx$$

$$= \int_\pi^0 x\,dx = -\frac{\pi^2}{2}$$

$$\therefore \text{原式} = -\frac{4}{9}+\frac{\pi^2}{2}$$

2. Gauss 公式

例：直接代入公式



$P=x^2+1$, $Q=-2y$, $R=3z$

$$I=\iiint_\Omega \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz$$

$$=\iiint_\Omega (2x+1)\,dxdydz$$

$$=V_\Omega + 2\int_0^1 x\,dx\int_0^{2-2x}dy\int_0^{\frac{2-2x-y}{2}}dz$$

$$=\frac{1}{3}\cdot\frac{1}{2}\cdot 2 + 2\cdot\int_0^1 x\left[(1-x)\cdot y-\frac{1}{4}y^2\right]\Big|_0^{2-2x}dx$$

$$=\frac{1}{3}+2\int_0^1 x(1-x^2)\,dx$$

$$=\frac{1}{3}+\frac{1}{6}$$

$$=\frac{1}{2}$$

3 Stokes公式

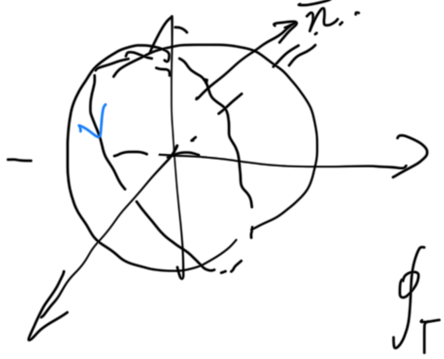
例：(1) $2\vec{i}+4\vec{j}+6\vec{k}$

(2) $\vec{i}+\vec{j}$

(3) $[x\sin(\cos z)-xy^2\cos(xz)]\vec{i}-y\sin(\cos z)\vec{j}+[y^2z\cos(xz)-x^2\cos y]\vec{k}$

补充：求 $\oint_{\Gamma} ydx+zdy+xdz$，其中 Γ 为圆周 $\begin{cases}x^2+y^2+z^2=a^2\\x+y+z=0\end{cases}$，若从x轴正向看

去，取逆时针方向为正向



取 Σ 为平面 $x+y+z=0$ 的上侧被 Γ 围成的部分

则 Σ 的面积为 πa^2. $dS=\pi a^2$

$n=(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3})$

$$\oint_{\Gamma} ydx+zdy+xdz$$

$$=\iint_{\Sigma}\begin{vmatrix}\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y&z&x\end{vmatrix}dS$$

$$=\iint_{\Sigma}\left(-\frac{1}{\sqrt3}-\frac{1}{\sqrt3}-\frac{1}{\sqrt3}\right)dS$$

$$=-\sqrt3\iint_{\Sigma}dS$$

$$=-\sqrt3\pi a^2$$

习题. 1. $\iint_D f(x,y)\,dxdy = \iint_D g(x)h(y)\,dxdy$

$$= \int_a^b dx \int_c^d g(x)h(y)\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy = RHS$$

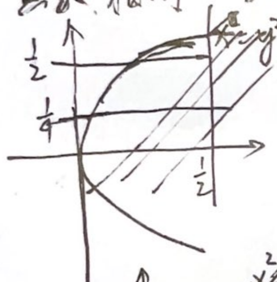
法二: $\left[\int_a^b f(x)\,dx\right]^2 = \int_a^b f(x)\,dx \int_a^b f(y)\,dy = \iint_D f(x)f(y)\,d\sigma$.

(例题)

$D = \{(x,y) \mid a \le x \le b, a \le y \le b\}$

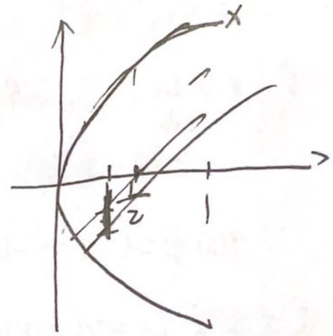
2. 看起来很熟悉的和可合并？

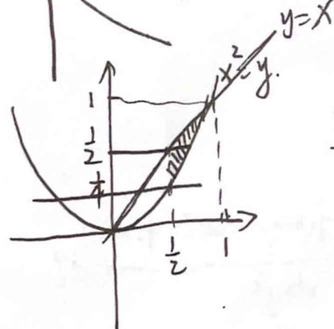
被积相同 → 合并积分区域



$$\int_{\frac14}^{\frac12} dy \int_{\frac12}^{\sqrt{y}} e^{\frac{y}{x}}\,dx$$

$$\int_{\frac12}^{1} dy \int_{y}^{\sqrt{y}} e^{\frac{y}{x}}\,dx$$

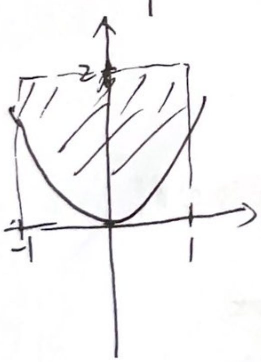




$$I = \iint_D e^{\frac{y}{x}}\,d\sigma = \int_{\frac12}^{1} dx \int_{x^2}^{x} e^{\frac{y}{x}}\,dy$$

$$= \int_{\frac12}^{1} (e - e^{x})\cdot x\,dx = \frac18(3e - 4\sqrt{e})$$

3.



$$\iint_D |y - x^2|\,d\sigma = 2\left(\iint_{D_1} (y - x^2)\,d\sigma + \iint_{D_1} (x^2 - y)\,d\sigma\right)$$

分别计算即可.

$$= \frac{46}{15}$$

4.

$$\iint_{[0,R]\times[0,R]} e^{-(x^2+y^2)}\,dxdy \geq \iint_{x^2+y^2\le R^2} e^{-(x^2+y^2)}\,dxdy$$

$$= \iint_{r\le R} e^{-r^2}\cdot r\,dr\,d\theta \qquad \le \iint_{x^2+y^2\le 2R^2} e^{-(x^2+y^2)}\,dxdy$$

同理, $\le \iint_{r^2\le 2R^2} e^{-r^2}\cdot r\,dr\,d\theta$

$R \to +\infty$ 两边夹

$\Rightarrow$ 原式 $= \frac{\pi}{4}$

[注: 可由此推出 $\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$]

$\downarrow$

原本积不出来

1. 设 $P = -\frac{yf(x,y)}{x^2+y^2}$，$Q = \frac{xf(x,y)}{x^2+y^2}$.

$$\frac{\partial Q}{\partial x} = \frac{xf_1'(x,y)}{x^2+y^2} - \frac{(x^2-y^2)f(x,y)}{(x^2+y^2)^2}, \quad \frac{\partial P}{\partial y} = \frac{(x^2-y^2)f(x,y)}{(x^2+y^2)^2} - \frac{yf_1'(x,y)}{x^2+y^2}$$

∴由Green公式 $\iint_{D_\varepsilon} \frac{xf_1'(x,y)+yf_2'(x,y)}{x^2+y^2}\,dxdy = \int_{\partial D_\varepsilon} Pdx+Qdy$

设 $L_\varepsilon: x^2+y^2=\varepsilon^2$

∂D_ε 包含 $\underset{\text{正向}}{\underset{\downarrow}{L_1}}$ 和 $\underset{\text{负向}}{\underset{\downarrow}{L_\varepsilon}}$ 两部分

∵ $f(x,y)=0\ (x^2+y^2=1)$. ∴ $\int_{L_1} Pdx+Qdy=0$.

求出曲线 L_ε 的参数方程：$\begin{cases} x=\varepsilon\cos t \\ y=\varepsilon\sin t \end{cases}$，$-\pi < t \le \pi$

$$∴ \int_{L_\varepsilon} Pdx+Qdy = \int_{-\pi}^{\pi}\left(P\frac{dx}{dt}+Q\frac{dy}{dt}\right)dt = \int_{-\pi}^{\pi}\frac{\varepsilon f(x,y)(x\cos t+y\sin t)}{x^2+y^2}dt$$

$$= \int_{-\pi}^{\pi} f(\varepsilon\cos t, \varepsilon\sin t)\,dt$$

$$∴ 原式 = \lim_{\varepsilon\to0^+}\int_{\partial D_\varepsilon} Pdx+Qdy = \lim_{\varepsilon\to0^+}\left(-\int_{L_\varepsilon}Pdx+Qdy\right) = -\int_{-\pi}^{\pi} f(0,0)\,dt$$

$$= -2\pi f(0,0) \qquad \cancel{= 2\pi f(0,0)}$$